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Supersymmetric non-abelian Born-Infeld revisited

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Abstract: We determine the non-abelian Born-Infeld action, including fermions, as it results from the four-point tree-level open superstring scattering amplitudes at order α'^2. We find that, after an appropriate field redefinition all terms at this order can be written as a symmetrised trace. We confront this action with the results that follow from kappa-symmetry and conclude that the recently proposed non-abelian kappa-symmetry cannot be extended to cubic orders in the Born-Infeld curvature.

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Contents

1. Introduction

One of the unsolved questions of D-brane physics concerns the form of the (tree-level) effective action for N coinciding D-branes beyond the leading term which is just U(N) super Yang-Mills theory. For a single D-brane, $N = 1$, the higher-order corrections are captured by (a supersymmetric version of) the Born-Infeld lagrangian [1, 2]. Once several D-branes are present, things become involved. On the one hand, the gauge field A_μ is non-abelian [3] and one has to give an ordering prescription for the higher-order terms. On the other hand, for Dp-branes, there are also $9 - p$ embedding coordinates X^i which are U(N) valued as well, and all background fields will depend on them. This is bound to be quite complicated. As a first step, many papers concentrated on D9-branes in order to avoid this second difficulty. Of course, the D9-brane action is closely related to the open superstring effective action with U(N) Chan-Paton factors.

The most direct way to obtain the effective action goes through the calculation of open string scattering amplitudes. This program yielded the purely bosonic terms through order $\alpha'^2 F^4$ [4, 5]. The full order α'^2 action in the abelian case was determined by [6]. It is obvious that at higher orders the complexity of this approach considerably increases. Several alternative techniques have been developed precisely with the aim to avoid these complications.

The most obvious alternative uses β-function calculations. This method proved extremely succesful in the abelian case: it was used to show that the (bosonic) Born-Infeld action is the effective action for the open superstring theory to all orders [5]. However, in the non-abelian case it becomes as unpleasant as the previous approach.

This led to the development of several, more indirect ways of attacking the problem. Some of them use supersymmetry as a guideline as the supersymmetry algebra in 10 dimensions is severely restricted. One obvious choice would be to use linear supersymmetry. This was exploited in [7, 8]. In particular, the work of [8] led to a full proposal for the effective action through order α'^2 including fermionic and derivative terms. The presence of a non-linearly realized supersymmetry provided some checks on this result and obviously raises the question whether there exists an underlying κ-invariant action. In the abelian case the answer is affirmative. In fact κ-symmetry gave the first explicit supersymmetrization of the abelian Born-Infeld action in a flat background [2].

In [9] the issue of κ-symmetry in the non-abelian case was addressed. Starting from a concrete ansatz, which was motivated by the abelian calculation, this resulted in a κ-invariant action including all terms quadratic in the field strengths up to quartic fermions.

A perhaps closely related approach uses the existence of BPS-type solutions [10]. While this method does not give any information on the fermionic terms, it does provide a powerful method to reconstruct the purely bosonic part of the action. In the abelian case it shows that the Born-Infeld action is unique. The extension to the non-abelian case is presently under study and will give information on the purely bosonic terms through order α'^4 including higher-order derivatives [10].

In the context of string theory, configurations involving constant magnetic background fields correspond, after T-duality, to D-branes at angles. The latter picture allows for a direct calculation of the spectrum which can then be compared to the spectrum as calculated from the non-abelian Born-Infeld action [11, 12]. Again, this program so far was restricted to the study of the bosonic terms only and partially fixed the effective action through order $\alpha'^4 F^6$ [13].

Finally, the Seiberg-Witten map might give further clues about the structure of the higher-order derivative terms [14].

A major issue in the construction of the effective action in the non-abelian case is the ordering of the fields. String theory unambiguously determines the $\alpha'^2 F^4$ terms to be a symmetrised trace. Modulo effects arising from higher-order derivative

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terms, this led Tseytlin to the conjecture that the full non-abelian Born-Infeld action should be defined through the symmetrised trace [15]. Soon thereafter this proposal was probed by comparing fluctuation spectra with those of the corresponding D-brane configurations and the result disagreed from order $\alpha'^4 F^6$ on [11, 12]. These results concerned bosonic terms only and one might wonder whether fermionic terms at order α'^2 already deviate from the symmetrised trace prescription. In [9] it was claimed that such a deviation indeed occurs. The claim of [9] was based on the assumption that a non-abelian generalization of κ-symmetry exists. Recent results in [8] indicate that the symmetrised trace prescription still holds for these fermionic terms at order α'^2.

To settle this issue, we will calculate in section 2 all terms in the effective action, including fermions, which can be determined from four-point string scattering amplitudes of order α'^2. We find that the string effective action, at this order and after a certain field redefinition, takes the form of a symmetrised trace. Furthermore, it agrees with the results in [8]. As we will discuss in section 3, we conclude that the non-abelian κ-symmetry as introduced in [9] does not work when cubic orders in the field strength F are included in the variation of the action. Nevertheless, the presence of a nonlinear supersymmetry in [8] suggests the existence of a different formulation, perhaps related to κ-symmetry, in which both supersymmetries arise after an appropriate gauge fixing.

2. Effective action from the string amplitudes

In this section we will summarize the computation of all the tree-level open string (disc) four-point amplitudes between the massless gauge bosons and their fermionic partners (gauginos). There is a 4 boson, a 4 fermion and a 2 boson / 2 fermion amplitude. We will call the external momenta $k_1, \ldots k_4$ (all taken as incoming), assign Chan-Paton labels $a, b, c, d = 1, \ldots, \dim \mathrm{U}(N)$, and wave-functions u_i to the external fermions and polarisations ϵ_j to the external bosons. This is depicted in figure 1 for the example of a 2 boson / 2 fermion amplitude. Our conventions as well as various useful identities are summarised in the appendix.

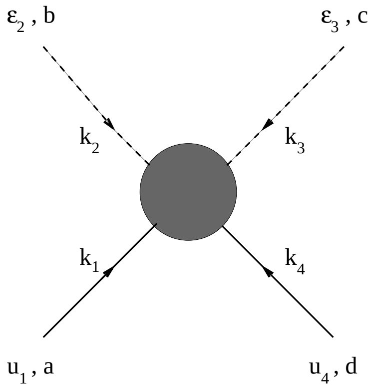

Figure 1: 2 boson / 2 fermion scattering amplitude.

2.1 The string amplitudes

Any of the 4 point amplitudes is a sum of six disc diagrams corresponding to the 6 different cyclic orderings of the vertex operators as shown in figure 2.

The contribution of each of the six orderings then is given [16, 17] by the product of

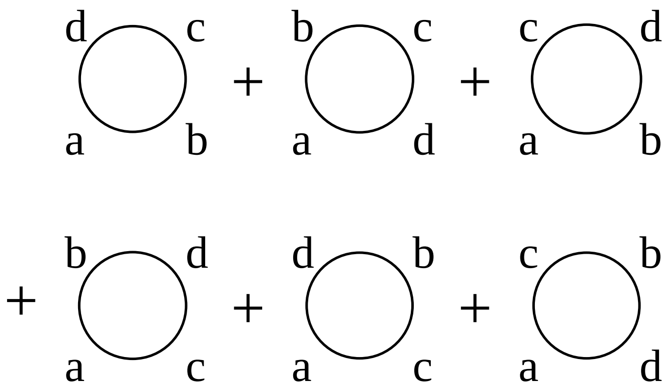

Figure 2: The six different cyclic orderings.

1. a trace of the product of matrices λ_a in the fundamental representation of $\mathrm{U}(N)$, taken in the cyclic order given by the diagram of figure 2, e.g. for the first one: $\mathrm{tr}\, \lambda_a \lambda_b \lambda_c \lambda_d \equiv t_{abcd}$

2. a function G depending on the two Mandelstam variables "flowing" through the diagram "horizontally" and "vertically". For the first diagram of figure 2 e.g. the vertical momentum flow gives $(k_1+k_2)^2 = s$ while the horizontal momentum flow gives $(k_1+k_4)^2 = u$. Clearly, the 1. and 2. diagram give $G(s,u)$, the 3. and 4. give $G(s,t)$ and the 5. and 6. give $G(t,u)$. The function G is given by

$$G(s,t) = \alpha'^2 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s - \alpha' t)} = \frac{1}{st} - \frac{\pi^2 \alpha'^2}{6} + \mathcal{O}(\alpha'^3) \tag{2.1}$$

   and is the same independent of the nature (boson or fermion) of the massless external states.

3. a kinematic factor K depending on the polarisations and wave-functions in the given cyclic order as well as on the momenta. It is independent of α'. This factor would actually be the same also for loop amplitudes. In the present example of 2 boson / 2 fermion scattering of figure 1, the 3. diagram of figure 2 would e.g. come with a $K(u_1, \epsilon_2, u_4, \epsilon_3)$.

4. a normalisation factor which we will take to be $-8ig^2$.

5. a minus sign for any diagram in figure 2 which differs from the first one by the permutation of two fermions. Note that these signs will be cancelled in the end by the corresponding antisymmetry of the K-factor.

Let us now discuss these kinematical factors K. They are given in ref. [16], where references to the original literature can be found. Some care has to be exercised while copying the formulas since our conventions are different from those of ref. [16]. The differences are: a) $s_{\mathrm{GSW}} = -s$, $t_{\mathrm{GSW}} = -u$, $u_{\mathrm{GSW}} = -t$, b) $\{\Gamma^\mu, \Gamma^\nu\}_{\mathrm{GSW}} = -2\eta^{\mu\nu}_{\mathrm{GSW}}$

while we take $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and c) we also must change the overall normalisation by a factor $-1/4$ for the 4 fermion and the 2 boson / 2 fermion case, while in the 4 boson case the GSW normalisation is appropriate.

For 4 bosons we get:

$$\begin{aligned} K(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = & -\frac{tu}{4}\epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot \epsilon_4 - \frac{su}{4}\epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4 - \frac{st}{4}\epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot \epsilon_3 - \\ & -\frac{s}{2}\mathcal{K}_s - \frac{t}{2}\mathcal{K}_t - \frac{u}{2}\mathcal{K}_u\,, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{K}_s &= \epsilon_1 \cdot k_4 \epsilon_3 \cdot k_2 \epsilon_2 \cdot \epsilon_4 + \epsilon_2 \cdot k_3 \epsilon_4 \cdot k_1 \epsilon_1 \cdot \epsilon_3 + \epsilon_1 \cdot k_3 \epsilon_4 \cdot k_2 \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_1 \epsilon_1 \cdot \epsilon_4 \\ \mathcal{K}_t &= \mathcal{K}_s|_{2\leftrightarrow 3} \\ \mathcal{K}_u &= \mathcal{K}_s|_{2\leftrightarrow 4}\,. \end{aligned} \tag{2.3}$$

Note that $K(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ is completely symmetric under any permutation $i \leftrightarrow j$ and it vanishes if we replace ϵ_i by k_i as required by gauge invariance.

For four fermions the K-factor is given by

$$K(u_1, u_2, u_3, u_4) = \frac{s}{8}\, \overline{u}_1 \gamma_\mu u_4\, \overline{u}_2 \gamma^\mu u_3 - \frac{u}{8}\, \overline{u}_1 \gamma_\mu u_2\, \overline{u}_4 \gamma^\mu u_3\,. \tag{2.4}$$

The u_i are the (commuting) ten-dimensional Majorana-Weyl fermion wave-functions. Hence we have $\overline{u}_i \gamma^\mu u_j = \overline{u}_j \gamma^\mu u_i$ and the Fierz identity

$$\overline{u}_1 \gamma_\mu u_2\, \overline{u}_3 \gamma^\mu u_4 + \overline{u}_1 \gamma_\mu u_3\, \overline{u}_4 \gamma^\mu u_2 + \overline{u}_1 \gamma_\mu u_4\, \overline{u}_2 \gamma^\mu u_3 = 0 \tag{2.5}$$

which together with the relation $s + t + u = 0$ implies that $K(u_1, u_2, u_3, u_4)$ is completely antisymmetric under the exchange of any two fermions, e.g. we have $K(u_1, u_2, u_4, u_3) = -K(u_1, u_2, u_3, u_4)$ etc.

For two fermions and two bosons, ref. [16] considers two cases separately: the two fermions are adjacent or not. Both cases actually lead to the *same* K-factor:

$$K(u_1, \epsilon_2, \epsilon_3, u_4) = K(u_1, \epsilon_2, u_4, \epsilon_3) = \frac{u}{8}A + \frac{s}{8}B \tag{2.6}$$

where we define the convenient expressions ($\not{k} \equiv k_\mu \gamma^\mu$)

$$\begin{aligned} A &= \overline{u}_1 \not{\epsilon}_2 (\not{k}_3 + \not{k}_4) \not{\epsilon}_3 u_4 \\ B &= 2\overline{u}_1 \left(\not{\epsilon}_3\, k_3 \cdot \epsilon_2 - \not{\epsilon}_2\, k_2 \cdot \epsilon_3 - \not{k}_3\, \epsilon_2 \cdot \epsilon_3\right) u_4\,. \end{aligned} \tag{2.7}$$

Using the on-shell properties $k_2 \cdot \epsilon_2 = k_3 \cdot \epsilon_3 = \not{k}_4 u_4 = \overline{u}_1 \not{k}_1 = 0$ one easily shows

$$\begin{aligned} A|_{2\leftrightarrow 3} &= A - B\,, & A|_{1\leftrightarrow 4} &= B - A \\ B|_{2\leftrightarrow 3} &= -B\,, & B|_{1\leftrightarrow 4} &= B\,, \end{aligned} \tag{2.8}$$

so that they are symmetric under the exchange of the two bosons and antisymmetric under exchange of the two fermions.

These kinematical factors are actually determined by the required (anti)symmetry, (linearized) gauge invariance and dimensional considerations.

It follows that any of the four-point (tree-level) amplitudes we are interested in takes the form

$$\begin{aligned} A_4 = &-8ig^2\ K(1,2,3,4)\times \\ &\times\left\{(t_{abcd}+t_{dcba})G(s,u)+(t_{abdc}+t_{cdba})G(s,t)+(t_{acbd}+t_{dbca})G(t,u)\right\}. \end{aligned} \tag{2.9}$$

Note that any minus signs introduced when two fermions in figure 2 are permuted with respect to the reference configuration has been cancelled by another minus sign when performing the same permutation on the arguments of K to rewrite it as $K(1,2,3,4)$.

Turning to the traces, they come in 3 combinations:

$$T_1 = t_{abcd}+t_{dcba} = \frac{1}{2}\left(d_{abe}d_{cde}+d_{ade}d_{bce}-d_{ace}d_{bde}\right) \tag{2.10}$$

$$T_2 = t_{abdc}+t_{cdba} = \frac{1}{2}\left(d_{abe}d_{cde}+d_{ace}d_{bde}-d_{ade}d_{bce}\right) \tag{2.11}$$

$$T_3 = t_{acbd}+t_{dbca} = \frac{1}{2}\left(d_{ace}d_{bde}+d_{ade}d_{bce}-d_{abe}d_{cde}\right). \tag{2.12}$$

where d_{abc} is given by $\{\lambda_a,\lambda_b\}=d_{abc}\lambda_c$. Properties of the d and f tensors are given in the appendix. Note that the symmetrised trace is given by

$$\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d = \frac{1}{12}\left(d_{abe}d_{cde}+d_{ace}d_{bde}+d_{ade}d_{bce}\right). \tag{2.13}$$

Inserting the α'-expansion of the G-function into (2.9) we get for any of the four-point amplitudes

$$A_4 = -8ig^2\ K(1,2,3,4)\sum_{n=0}^{\infty} a_4^{(n)}\alpha'^n. \tag{2.14}$$

The lowest order term can be written in 3 equivalent ways:

$$\begin{aligned} a_4^{(0)} &= \frac{1}{s}\left(\frac{1}{t}f_{ace}f_{bde}+\frac{1}{u}f_{ade}f_{bce}\right) = -\frac{1}{u}\left(\frac{1}{s}f_{abe}f_{cde}+\frac{1}{t}f_{ace}f_{bde}\right) \\ &= \frac{1}{t}\left(\frac{1}{s}f_{abe}f_{cde}-\frac{1}{u}f_{ade}f_{bce}\right). \end{aligned} \tag{2.15}$$

This vanishes in the abelian case: there is no lowest order photon-photon scattering. Clearly, there is no order α' contribution and $a_4^{(1)}=0$.

The obvious fact about the order α'^2 contribution is that it is always a symmetrised trace. Indeed, at order α'^2 the function G is just a constant, and thus all traces contribute equally, leading to a symmetrised trace:

$$a_4^{(2)} = -\pi^2\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\,. \tag{2.16}$$

Clearly, there is no reason for any other $a_4^{(n)}$ to be a symmetrised trace. Note that nevertheless, by construction, all $a_4^{(n)}$ are completely symmetric under exchange of any two external states, so that the symmetry properties of the amplitude are correctly given by those of the kinematical factors $K(1,2,3,4)$.

For convenience of comparison with the field theory amplitudes, we explicitly write down the amplitudes up to and including the order α'^2 terms:
four bosons

$$\begin{aligned}A_4^{4\mathrm{b}} = &\left[4ig^2 \frac{1}{s}\left(\mathcal{K}_t - \mathcal{K}_u + \frac{u-t}{4}\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4\right) f_{abe}f_{cde} - \right.\\ &\left. - ig^2\,\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4\,(f_{ace}f_{bde} + f_{ade}f_{bce})\right] + [2\leftrightarrow 3, b\leftrightarrow c] + [2\leftrightarrow 4, b\leftrightarrow d] + \\ &+ 8ig^2\pi^2\alpha'^2 K(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d + \mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{2.17}$$

two bosons and two fermions

$$\begin{aligned}A_4^{2\mathrm{b}/2\mathrm{f}} &= -ig^2\,\frac{B}{u}\,f_{ade}f_{bce} + ig^2\,\frac{A}{s}\,f_{abe}f_{cde} + ig^2\,\frac{A-B}{t}\,f_{ace}f_{bde} + \\ &\quad + ig^2\pi^2\alpha'^2\,(uA + sB)\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d + \mathcal{O}(\alpha'^3)\end{aligned} \tag{2.18}$$

four fermions (using the Fierz identity)

$$\begin{aligned}A_4^{4\mathrm{f}} = &\left[-ig^2\,\overline{u}_1\gamma^\mu u_2\,\overline{u}_4\gamma_\mu u_3\,\frac{1}{s}f_{abe}f_{cde} + ig^2\,\overline{u}_1\gamma^\mu u_3\,\overline{u}_4\gamma_\mu u_2\,\frac{1}{t}f_{ace}f_{bde} - \right.\\ &\left. - ig^2\,\overline{u}_1\gamma^\mu u_4\,\overline{u}_2\gamma_\mu u_3\,\frac{1}{u}f_{ade}f_{bce}\right] + \\ &+ ig^2\pi^2\alpha'^2\,(s\,\overline{u}_1\gamma^\mu u_4\,\overline{u}_2\gamma_\mu u_3 - u\,\overline{u}_1\gamma^\mu u_2\,\overline{u}_4\gamma_\mu u_3)\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d + \mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{2.19}$$

2.2 The ansatz for the effective action

2.2.1 The α' expansion

Our goal is to find the effective action which reproduces the α'-expansion of the open superstring four-point amplitude of the previous subsection. At lowest order in α' this is of course well-known to be the U(N) $\mathcal{N}=1$ super Yang-Mills theory in ten dimensions

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\chi}\gamma^\mu D_\mu\chi\right)\,. \tag{2.20}$$

This fixes the normalisations of the fields. The main effort in this section will be devoted to (almost) uniquely determining the effective action at order α'^2. Of course the action will be determined only up to terms that vanish “on shell”. Since we look at four-point amplitudes we in principle fix all terms of the form F^4, $F^2\chi^2$ and χ^4 including all higher order derivatives.

The possible terms at order α'^2 are given by dimensional analysis: in any space-time dimension, dimensionless quantities are $\alpha' g F_{\mu\nu}$ and $\alpha'^2 g^2\,\overline{\chi}\gamma D\chi$ where g is the Yang-Mills coupling constant.

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2.2.2 The abelian Born-Infeld action

For the sake of comparison we now give the expansion of the *abelian* Born-Infeld action [2]:

$$
\begin{aligned}
\mathcal{L}_{\rm BI} &= \frac{1}{\tilde{\alpha}'^2}\left\{1-\left[\det\left(\eta^\mu{}_\nu+\tilde{\alpha}'F^\mu{}_\nu-i\tilde{\alpha}'^2\,\overline{\chi}\gamma^\mu\partial_\nu\chi-\frac{\tilde{\alpha}'^4}{4}\,\overline{\chi}\gamma^\rho\partial^\mu\chi\,\overline{\chi}\gamma_\rho\partial_\nu\chi\right)\right]^{1/2}\right\}\\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}\overline{\chi}\partial\!\!\!/\chi+\frac{i}{2}\tilde{\alpha}'\,\overline{\chi}\gamma_\mu\partial_\nu\chi F^{\mu\nu}+\\
&\quad+\frac{\tilde{\alpha}'^2}{8}\left(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}-\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right)+\frac{i}{2}\tilde{\alpha}'^2\overline{\chi}\gamma_\mu\partial_\nu\chi(F^{\mu\rho}F_\rho{}^\nu-\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma})+\\
&\quad+\frac{\tilde{\alpha}'^2}{4}\left(\frac{1}{2}\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\mu\partial_\nu\chi-\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\nu\partial_\mu\chi+\frac{1}{2}\left(\overline{\chi}\partial\!\!\!/\chi\right)^2\right)+\mathcal{O}(\tilde{\alpha}'^3)\,, \qquad (2.21)
\end{aligned}
$$

where

$$\tilde{\alpha}'=2\pi g\alpha'\,. \qquad (2.22)$$

Note that our present computation of four-point amplitudes at order α'^2 will not be sensitive to terms of the form $\tilde{\alpha}'^2(\overline{\chi}\partial\!\!\!/\chi)^2$ and $\tilde{\alpha}'^2\overline{\chi}\partial\!\!\!/\chi F_{\rho\sigma}F^{\rho\sigma}$ as they vanish on-shell. Nevertheless, they could be determined from higher-point amplitudes. Henceforth we will drop such terms. A similar remark applies to the order $\tilde{\alpha}'$ term $\overline{\chi}\gamma_\mu\partial_\nu\chi F^{\mu\nu}$ which upon partial integration and using the Majorana properties can be written as $A^\mu\overline{\chi}(\gamma_\mu\partial^2-\partial_\mu\partial\!\!\!/)\chi$ which vanishes for on-shell fermions. Thus this term does not contribute to a three-point amplitude, but it gives a non-vanishing contribution to the 2 fermion / 2 boson four-point amplitude via a one-particle reducible diagram with an internal fermion line.

This order α' term in the abelian Born-Infeld action can be removed by the field redefinition

$$\chi\to\chi+\frac{1}{4}\tilde{\alpha}'F_{\rho\sigma}\gamma^{\rho\sigma}\chi \qquad (2.23)$$

at the expense of modifying the order α'^2 terms. Dropping all terms involving $\tilde{\alpha}'^2\partial\!\!\!/\chi$ we then get

$$
\begin{aligned}
\mathcal{L}'_{\rm BI}|_{\rm on-shell} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}\overline{\chi}\partial\!\!\!/\chi+\frac{\tilde{\alpha}'^2}{8}\left(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}-\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right)+\\
&\quad+\frac{i}{4}\tilde{\alpha}'^2\,\overline{\chi}\gamma_\mu\partial_\nu\chi F^{\mu\rho}F_\rho{}^\nu-\frac{i}{8}\tilde{\alpha}'^2\overline{\chi}\gamma_{\mu\nu\rho}\partial_\sigma\chi F^{\mu\nu}F^{\rho\sigma}+\\
&\quad+\frac{\tilde{\alpha}'^2}{8}\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\mu\partial_\nu\chi-\frac{\tilde{\alpha}'^2}{4}\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\nu\partial_\mu\chi+\mathcal{O}(\tilde{\alpha}'^3)\,. \qquad (2.24)
\end{aligned}
$$

Note the modified coefficient of the $\overline{\chi}\gamma\partial\chi FF$-term and the new term involving $\gamma_{\mu\nu\rho}$.

There are two four-fermion terms, but they are related by a Fierz transformation:

$$\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\nu\partial_\mu\chi\simeq\frac{2}{3}\,\overline{\chi}\gamma^\mu\partial^\nu\chi\,\overline{\chi}\gamma_\mu\partial_\nu\chi\,, \qquad (2.25)$$

where $\simeq$ means equality up to on-shell terms. This is most easily seen to follow from (A.6) by setting $\psi = \partial_\nu \chi$, $\lambda = \partial_\mu \chi$ and $\varphi = \chi$: dropping on-shell terms and using also (A.5) this becomes

$$\begin{aligned}\overline{\chi}\gamma^\mu\partial_\nu\chi\partial_\mu\overline{\chi}\gamma^\nu\chi &\simeq \frac{1}{8}\overline{\chi}\gamma^{\rho\sigma\mu}\chi\ \partial_\mu\overline{\chi}\gamma_{\nu\rho\sigma}\partial^\nu\chi - \frac{1}{48}\overline{\chi}\gamma^{\rho\sigma\lambda}\chi\ \partial_\mu\overline{\chi}\gamma_{\rho\sigma\lambda}\partial^\mu\chi \\ &\simeq -\overline{\chi}\gamma^\mu\partial^\nu\chi\ \overline{\chi}\gamma_\mu\partial_\nu\chi + \frac{1}{2}\overline{\chi}\gamma^\mu\partial^\nu\chi\ \overline{\chi}\gamma_\nu\partial_\mu\chi \end{aligned} \tag{2.26}$$

from which follows (2.25).

2.2.3 The ansatz for the non-abelian effective action

We write the effective action as

$$\mathcal{L} = \mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{4\mathrm{b}} + \mathcal{L}_{2\mathrm{b}/2\mathrm{f}} + \mathcal{L}_{4\mathrm{f}} + \mathcal{L}_* + \mathcal{O}(\alpha'^3 g^2, \alpha'^2 g^3) \tag{2.27}$$

with $\mathcal{L}_{4\mathrm{b}}$, $\mathcal{L}_{2\mathrm{b}/2\mathrm{f}}$ and $\mathcal{L}_{4\mathrm{f}}$ containing the order α' and α'^2 terms needed to reproduce the string amplitudes to this order. The piece $\mathcal{L}_*$ contains any terms $\sim \tilde{\alpha}'^2 \not{D}\chi$, $\sim \tilde{\alpha}'^2 D_\mu \overline{\chi}\gamma^\mu$, $\sim \tilde{\alpha}'^2 D_\mu F^{\mu\nu}$ that vanish on-shell and do not contribute to the four-point amplitudes as discussed above. In the following we write $\mathcal{L}_1 \simeq \mathcal{L}_2$ if $\mathcal{L}_1$ and $\mathcal{L}_2$ only differ up to terms in $\mathcal{L}_*$ and up to partial integration. The meaning of $\mathcal{O}(\alpha'^3 g^2, \alpha'^2 g^3)$ is the following: an n-point (tree) amplitude comes with a factor g^{n-2} and neglecting $\mathcal{O}(\alpha'^2 g^3)$ terms is tantamount to not taking into account terms that only contribute to five- and higher-point amplitudes. On the other hand, $\mathcal{O}(\alpha'^3 g^2)$ terms arise from four-point amplitudes but contain more derivatives and will not be considered here either.

The *purely bosonic piece* $\mathcal{L}_{4\mathrm{b}}$ is well-established:

$$\mathcal{L}_{4\mathrm{b}} = \tilde{\alpha}'^2\, \mathrm{str}\left(\frac{1}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{32}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right) \tag{2.28}$$

where $\tilde{\alpha}' = 2\pi g \alpha'$. Since this contains exactly four F's, the contribution to the four gluon amplitude is obtained by extracting the interaction where each $F^a_{\mu\nu}$ is replaced simply by $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$. There is then a single order α'^2 four gluon vertex contributing to the amplitude, and it is a straightforward exercise to show that the result coincides with the order α'^2 part of the string amplitude $A_4^{4\mathrm{b}}$ in (2.17). In fact, it is not necessary to check all the terms in $K(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ since the structure of K is fixed by gauge invariance and permutation symmetry. It is e.g. enough to check that (2.28) yields the $\epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot \epsilon_4$ term with the correct coefficient. It is also easy to show that eq. (2.28) with the symmetrised trace is the *unique* interaction that reproduces the string amplitude at this order.

More interesting is the *mixed piece* $\mathcal{L}_{2\mathrm{b}/2\mathrm{f}}$. Taking into account the Majorana-Weyl properties (see appendix), we find that a general ansatz for the non-abelian effective action at orders α' and α'^2 is

$$\begin{aligned}\mathcal{L}_{2b/2f} &= ic_1\tilde\alpha' d_{abc}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\nu} + i\tilde\alpha'^2\omega_{abcd}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\rho}F^{d}_{\ \rho}{}^{\nu} + \\ &\quad + i\tilde\alpha'^2\xi_{abcd}\overline{\chi}^a\gamma_{\mu\nu\rho}\left(D_\sigma\chi^b F^{c\mu\nu}F^{d\rho\sigma} - D^\rho\chi^b F^{c\mu\sigma}F^{d}_{\ \sigma}{}^{\nu}\right) \\ &\simeq ic_1\tilde\alpha' d_{abc}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\nu} + i\tilde\alpha'^2 y_{abcd}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\rho}F^{d}_{\ \rho}{}^{\nu} + \\ &\quad + i\tilde\alpha'^2\xi_{abcd}\overline{\chi}^a\gamma_\mu\gamma_\nu\gamma_\rho D_\sigma\chi^b F^{c\mu\nu}F^{d\rho\sigma}\,,\end{aligned} \tag{2.29}$$

with

$$y_{abcd} = \omega_{abcd} - \xi_{abcd} - \xi_{abdc}\,. \tag{2.30}$$

We have not specified the gauge structure of the order $\tilde\alpha'^2$ terms: ω_{abcd} and ξ_{abcd} are arbitrary so far.[1] On the other hand, for the order $\tilde\alpha'$ term we have specified $d_{abc} = \operatorname{str}\lambda_a\lambda_b\lambda_c$. The only other possibility would be f_{abc}. In this latter case however, $f_{abc}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\nu} \simeq -\frac{1}{2}f_{abc}\overline{\chi}^a\gamma_\mu\chi^b D_\nu F^{c\mu\nu}$. In order to somewhat simplify our discussion we will assume from the outset that they are not present and we start with an action as given by (2.29).

As in the abelian case, the order $\tilde\alpha'$ term does not contribute to a three-point amplitude between on-shell states, which is consistent with the absence of such an amplitude in string theory. It is convenient to first eliminate this order α' term by performing a field redefinition and then to compute the amplitude. So we let

$$\chi^a \to \chi^a + \frac{c_1}{2}\tilde\alpha'\, d_{abc}F^b_{\rho\sigma}\gamma^{\rho\sigma}\chi^c\,. \tag{2.31}$$

This will not change $\mathcal{L}_{4b}$ or $\mathcal{L}_{4f}$ but it will affect $\mathcal{L}_{2b/2f}$ which becomes (up to on-shell terms and total derivatives)

$$\begin{aligned}\mathcal{L}'_{2b/2f} &= i\tilde\alpha'^2 y_{abcd}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\rho}F^{d}_{\ \rho}{}^{\nu} + i\tilde\alpha'^2 z_{abcd}\overline{\chi}^a\gamma_\mu\gamma_\nu\gamma_\rho D_\sigma\chi^b F^{c\mu\nu}F^{d\rho\sigma} \\ &= i\tilde\alpha'^2\widetilde{y}_{abcd}\overline{\chi}^a\gamma_\mu D_\nu\chi^b F^{c\mu\rho}F^{d}_{\ \rho}{}^{\nu} + i\tilde\alpha'^2 z_{abcd}\overline{\chi}^a\gamma_{\mu\nu\rho}D_\sigma\chi^b F^{c\mu\nu}F^{d\rho\sigma}\end{aligned} \tag{2.32}$$

with

$$\begin{aligned}z_{abcd} &= \xi_{abcd} - \frac{c_1^2}{2}d_{ace}d_{bde}\,, \\ \widetilde{y}_{abcd} &= y_{abcd} + 2z_{abcd} = \omega_{abcd} + \xi_{abcd} - \xi_{abdc} - c_1^2 d_{ace}d_{bde}\,.\end{aligned} \tag{2.33}$$

It is clear from these relations that a symmetrised trace presciption can hold at best for $\mathcal{L}_{2b/2f}$ *or* $\mathcal{L}'_{2b/2f}$, but not both. Note that any part of y_{abcd} or $\widetilde{y}_{abcd}$ that is antisymmetric in a and b and symmetric in c and d, vanishes on shell by virtue of the Bianchi identity for F. Hence we can assume $y_{[ab](cd)} = \widetilde{y}_{[ab](cd)} = 0$

For the *four fermion interaction* $\mathcal{L}_{4f}$ we take the ansatz

$$\mathcal{L}_{4f} = \tilde\alpha'^2 g_{abcd}\overline{\chi}^a\gamma^\mu D^\nu\chi^b\,\overline{\chi}^c\gamma_\mu D_\nu\chi^d + \tilde\alpha'^2 h_{abcd}\overline{\chi}^a\gamma^\mu D^\nu\chi^b\,\overline{\chi}^c\gamma_\nu D_\mu\chi^d\,. \tag{2.34}$$

[1] The ansatz (2.29) in its present form still contains terms that vanish on-shell, and which really belong in $\mathcal{L}_*$. We will come back to this point after y and ξ have been matched to the string amplitude results.

Other terms could be written down e.g. $j_{abcd}\overline{\chi}^a\gamma^{\mu\nu\rho}D_\rho\chi^b\,\overline{\chi}^c\gamma_\mu D_\nu\chi^d$ or $l_{abcd}\overline{\chi}^a\gamma^{\mu\nu\rho}\times D^\sigma\chi^b\overline{\chi}^c\gamma_{\mu\nu\rho}D_\sigma\chi^d$. However, using Fierz identities, all of them can be rewritten as (2.34), up to on-shell terms. Similarly, one may assume that h_{abcd} is symmetric under interchange of a and b, or of c and d, or of ab and cd, and that

$$g_{abcd} = g_{cdab} \quad \text{and} \quad g_{(ab)[cd]} = g_{[ab](cd)} = 0 \qquad \Rightarrow \qquad g_{abcd} = g_{badc}\,. \tag{2.35}$$

Finally note that the only order α' term would be a new fermion bilinear like $\tilde{\alpha}'\overline{\chi}^a\gamma^{\mu\nu\rho}D_\mu D_\nu D_\rho\chi^a$. It would give an order α' two fermion - one gluon vertex and would contribute to the four fermion scattering via one-particle reducible gluon exchange diagrams, but this term actually reduces to the term discussed below (2.30).

2.3 Matching the amplitudes

The four gluon amplitude has already been discussed above.

2.3.1 Matching the 2 boson / 2 fermion amplitude

The most convenient form of the relevant interaction is the first line of (2.32), i.e. *after* the field redefinition (2.31). Indeed, (2.32) only contributes two terms to the 2 boson / 2 fermion interaction, obtained upon replacing $D_\lambda \to \partial_\lambda$ and $F^a_{\mu\nu} \to \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$. Obviously, there is no order α' piece, while the computation of the order α'^2 contribution to the amplitude is a bit lengthy but straightforward. We get:

$$\begin{aligned}
A_4^{2b/2f} = i\tilde{\alpha}'^2\Big\{ & A(tz^+ + sz^-) + \\
& + \overline{u}_1\not\epsilon_3 u_4\left[2k_1\cdot\epsilon_2(tz^+ + sz^-) + \frac{1}{2}(tk_1\cdot\epsilon_2 - sk_4\cdot\epsilon_2)(y_{dacb} + y_{adcb})\right] - \\
& - \overline{u}_1\not\epsilon_2 u_4\left[2k_4\cdot\epsilon_3(tz^+ + sz^-) + \frac{1}{2}(tk_4\cdot\epsilon_3 - sk_1\cdot\epsilon_3)(y_{dabc} + y_{adbc})\right] - \\
& - \overline{u}_1\not k_3 u_4\epsilon_2\cdot\epsilon_3\left[-2tz^+ + \frac{s}{2}(y_{adbc} + y_{dacb}) - \frac{t}{2}(y_{dabc} + y_{adcb})\right] + \\
& + \overline{u}_1\not k_3 u_4\Big[k_1\cdot\epsilon_2 k_1\cdot\epsilon_3(y_{dabc} - y_{dacb}) + k_4\cdot\epsilon_2 k_4\cdot\epsilon_3(y_{adcb} - y_{adbc}) - \\
& \qquad - k_1\cdot\epsilon_2 k_4\cdot\epsilon_3(4z^- + y_{adbc} + y_{dacb}) + \\
& \qquad + k_4\cdot\epsilon_2 k_1\cdot\epsilon_3(4z^+ + y_{dabc} + y_{adcb})\Big]\Big\}
\end{aligned} \tag{2.36}$$

with

$$z^+ = z_{dabc} + z_{adcb}\,, \qquad z^- = z_{adbc} + z_{dacb} \tag{2.37}$$

and where A (and B) were defined in (2.7). As a first check, note that this indeed vanishes if we replace $\epsilon_i \to k_i$, as required by gauge invariance.

As a further consistency check, note that we could have started with the interaction (2.29), i.e. *before* the field redefinition (2.31). Then the term $\sim y_{abcd}$ contributes as above in (2.36) while the contribution of the term $\sim \xi_{abcd}$ can also be read from (2.36) by replacing $z_{abcd} \to \xi_{abcd}$ and analogously $z^+ \to \xi^+ = \xi_{dabc} + \xi_{adcb}$ and $z^- \to \xi^- = \xi_{adbc} + \xi_{dacb}$. But now we have in addition the contributions from the order $\tilde{\alpha}'$ piece. This yields a new order $\tilde{\alpha}'$ cubic (2 fermion - 1 gluon) and a new quartic (2 fermion - 2 gluon) vertex. Thus we get various non-vanishing contributions to the 2 boson / 2 fermion amplitude at order $\tilde{\alpha}'$, but they sum up to zero

$$A_4^{\mathrm{2b,2f}}|_{\mathrm{order}\ \tilde{\alpha}'} = 0 \tag{2.38}$$

in agreement with the string amplitude. At order $\tilde{\alpha}'^2$ there are the two diagrams with an internal fermion in the s or t channel and both vertices being the cubic order $\tilde{\alpha}'$ interaction. Their sum yields

$$\begin{aligned} A_s^{\tilde{\alpha}'^2} + A_t^{\tilde{\alpha}'^2} \;=\; & -ic_1^2\tilde{\alpha}'^2\Big\{\left(A + 2k_1\cdot\epsilon_2\overline{u}_1\not{\epsilon}_3 u_4 - 2k_4\cdot\epsilon_3\overline{u}_1\not{\epsilon}_2 u_4\right)\left(td_{ace}d_{bde} + sd_{abe}d_{cde}\right) + \\ & + \overline{u}_1\not{k}_3 u_4\epsilon_2\cdot\epsilon_3 2td_{ace}d_{bde} + \\ & + \overline{u}_1\not{k}_3 u_4\left(4k_4\cdot e_2 k_1\cdot\epsilon_3 d_{ace}d_{bde} - 4k_1\cdot e_2 k_4\cdot\epsilon_3 d_{abe}d_{cde}\right)\Big\}\,. \end{aligned} \tag{2.39}$$

This has to be added to the contributions $\sim y_{abcd}$ and $\sim \xi_{abcd}$ as obtained from (2.36) as discussed above. Not too surprising, we find that the result of adding this contribution is just to shift

$$\xi^+ \to \xi^+ - c_1^2 d_{ace}d_{bde}\,, \qquad \xi^- \to \xi^- - c_1^2 d_{abe}d_{cde} \tag{2.40}$$

without affecting the y_{abcd}. This corresponds to

$$\xi_{abcd} \to \xi_{abcd} - \frac{c_1^2}{2}d_{ace}d_{bde} \tag{2.41}$$

which is nothing but replacing ξ_{abcd} by z_{abcd}. Thus in the end we get exactly the same result (2.36) as obtained from the interaction (2.32) *after* the field redefinition.

Matching the result (2.36) to the corresponding string amplitude (2.18) (recall that $\tilde{\alpha}' = 2\pi g\alpha'$) yields the following conditions

$$z^+ = z^- = -\frac{1}{4}\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\,, \qquad y_{adbc} + y_{dabc} = \mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\,, \qquad y_{adbc} = y_{adcb}\,. \tag{2.42}$$

This can be equivalently written as

$$z_{abcd} + z_{badc} = -\frac{1}{4}\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\,, \qquad y_{(ab)cd} = \frac{1}{2}\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\,, \qquad y_{ab[cd]} = 0\,. \tag{2.43}$$

Using the results of the appendix on the general form of 4-index tensors arising from a single trace, the most general solution is (recall $y_{[ab](cd)} = 0$)

$$\begin{aligned} y_{abcd} &= \frac{1}{2}\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d + y_5 d_{cde}f_{abe} \\ z_{abcd} &= -\frac{1}{8}\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d + z_4\, d_{abe}f_{cde} + z_5\, d_{cde}f_{abe} \end{aligned} \tag{2.44}$$

where y_5, z_4 and z_5 are undetermined parameters.

The presence of the undetermined parameters y_5, z_4 and z_5 is related to the fact that the tensors y and z were not a priori restricted to avoid the presence of contributions in the ansatz (2.29) that vanish on-shell. In fact, all three unknown parameters give contributions to the effective action that vanish on-shell, and can be eliminated by field redefinitions.

In the contribution y_5, one uses

$$\begin{aligned} y_5 d_{cde} f_{abe} \overline{\chi}^a \gamma_\mu D_\nu \chi^b F^{c\mu\rho} F^d{}_\rho{}^\nu &= \frac{1}{2} y_5 d_{cde} f_{abe} D_\nu (\overline{\chi}^a \gamma_\mu \chi^b) F^{c\mu\rho} F^d{}_\rho{}^\nu \\ &\simeq -\frac{1}{2} y_5 d_{cde} f_{abe} \overline{\chi}^a \gamma_\mu \chi^b (D_\nu F^{c\mu\rho}) F^d{}_\rho{}^\nu \\ &= +\frac{1}{4} y_5 d_{cde} f_{abe} \overline{\chi}^a \gamma_\mu \chi^b (D^\mu F^{c\rho}{}_\nu) F^d{}_\rho{}^\nu \simeq 0 \,, \end{aligned} \tag{2.45}$$

where the last step requires a partial integration. The contribution z_5 contains the same term as y_5, and in addition

$$z_5 d_{cde} f_{abe} \overline{\chi}^a \gamma_{\mu\nu\rho} D_\sigma \chi^b F^{c\mu\nu} F^{d\rho\sigma} \simeq -\frac{1}{4} z_5 d_{cde} f_{abe} \overline{\chi}^a \gamma_{\mu\nu\rho\sigma\lambda} D^\lambda \chi^b F^{c\mu\nu} F^{d\rho\sigma} \,, \tag{2.46}$$

where we have used the fact that the product of the two F's is, due to the symmetry in cd, completely antisymmetric in $\mu\nu\rho\sigma$. The expression then vanishes due to the Bianchi identity for F after partial integration. The contribution z_4 requires a cancellation between the two contributions in (2.32). The trick here is to write

$$2 d_{abe} f_{cde} \overline{\chi}^a \gamma_\mu D_\nu \chi^b F^{c\mu\rho} F^d{}_\rho{}^\nu \simeq -d_{abe} f_{cde} \overline{\chi}^a \gamma_{\mu\nu\lambda} D^\lambda \chi^b F^{c\mu\rho} F^d{}_\rho{}^\nu \tag{2.47}$$

and then to do a partial integration in both terms in (2.32). The cancellation occurs because

$$f_{cde} \left(D^{[\lambda} (F^{c\mu\sigma} F^d{}_\sigma{}^{\nu]}) - D_\sigma (F^{c[\mu\nu} F^{d\lambda]\sigma}) \right) \simeq 0 \tag{2.48}$$

where antisymmetrization is over the indices $\mu\nu\lambda$.

The matching of the four-point amplitude has therefore completely determined the 2 boson / 2 fermion part of the effective action.

2.3.2 Matching the 4 fermion amplitude

There are again two possible types of contributions to the four fermion amplitude: one-particle irreducible diagrams coming from the quartic interactions of $\mathcal{L}_{4\mathrm{f}}$, eq. (2.34), and, possibly, one-particle reducible gluon exchange diagrams using the cubic vertex from the order $\tilde{\alpha}'$ term in $\mathcal{L}_{2\mathrm{b}/2\mathrm{f}}$ before the field redefinition. This cubic vertex however vanishes if both fermions are on shell, so that these gluon exchange diagrams do not contribute to the 4 fermion amplitude. This is consistent with the fact that the field redefinition does not affect $\mathcal{L}_{4\mathrm{f}}$. In particular also, there is no order α' contribution to the amplitude.

We will now discuss the contributions of the two terms in $\mathcal{L}_{4\mathrm{f}}$ to the 4 fermion amplitude. We will argue soon that the second term in $\mathcal{L}_{4\mathrm{f}}$ cannot reproduce anything that looks like the string amplitude unless it can be transformed - using some Fierz identity - into a term with the same Lorentz index structure as the first one in $\mathcal{L}_{4\mathrm{f}}$. So we begin by examining the contribution of this first term alone. Obviously, its contribution to the amplitude contains $\overline{u}_1\gamma_\mu u_2\overline{u}_3\gamma^\mu u_4$, $\overline{u}_1\gamma_\mu u_4\overline{u}_2\gamma^\mu u_3$ and $\overline{u}_1\gamma_\mu u_3\overline{u}_2\gamma^\mu u_4$. Using the Fierz identity (2.5) this last expression can be rewritten as a combination of the two other, and, upon taking into account (2.35) we get

$$A_4^{4\mathrm{f}}|_{g-\mathrm{terms}} = -2i\tilde{\alpha}'^2\Big\{\left[(g_{acbd}+g_{adbc})\,s - g_{adcb}\,t - g_{acdb}\,u\right]\overline{u}_1\gamma_\mu u_4\overline{u}_2\gamma^\mu u_3 - \\ - \left[(g_{acdb}+g_{abdc})\,u - g_{abcd}\,t - g_{acbd}\,s\right]\overline{u}_1\gamma_\mu u_2\overline{u}_3\gamma^\mu u_4\Big\}\,. \tag{2.49}$$

Also in this case we will not attempt to restrict g a priori to avoid terms that vanish on-shell. Comparing with the string amplitude we find that, if and only if

$$g_{abcd} = g_{acbd}\,, \tag{2.50}$$

the amplitude reduces to the desired form

$$A_4^{4\mathrm{f}}|_{g-\mathrm{terms}} = -2i\tilde{\alpha}'^2\left(g_{acbd}+g_{abdc}+g_{adcb}\right)\left(s\;\overline{u}_1\gamma_\mu u_4\overline{u}_2\gamma^\mu u_3 - u\;\overline{u}_1\gamma_\mu u_2\overline{u}_3\gamma^\mu u_4\right). \tag{2.51}$$

The symmetry requirements (2.50) and (2.35) on g_{abcd} and the results of the appendix on 4-index tensors determine it to be of the form

$$g_{abcd} = \frac{g_1}{12}\left(d_{abe}d_{cde}+d_{ace}d_{bde}\right)+\frac{g_3}{12}d_{ade}d_{bce} \tag{2.52}$$

which in turn implies that the contribution of the first term in $\mathcal{L}_{4\mathrm{f}}$ to the amplitude can be written as

$$A_4^{4\mathrm{f}}|_{g-\mathrm{terms}} = -2i\tilde{\alpha}'^2(2g_1+g_3)\,\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d\left(s\;\overline{u}_1\gamma_\mu u_4\overline{u}_2\gamma^\mu u_3 - u\;\overline{u}_1\gamma_\mu u_2\overline{u}_3\gamma^\mu u_4\right). \tag{2.53}$$

Next, we consider the second term in $\mathcal{L}_{4\mathrm{f}}$. As discussed above, we may assume $h_{abcd} = h_{bacd} = h_{abdc}$. A straightforward computation shows that it contributes terms like $\overline{u}_1\not{k}_3u_2\,\overline{u}_3\not{k}_1u_4$ to the amplitude which are not of the desired form $\overline{u}_1\gamma_\mu u_2\overline{u}_3\gamma^\mu u_4$ etc. However, we will now show that if also $h_{dbca} = h_{abcd}$ then by a Fierz transformation these terms actually have the desired form. Note that requiring $h_{dbca} = h_{abcd}$ together with $h_{abcd} = h_{bacd} = h_{abdc}$ implies that h_{abcd} is completely symmetric in all its indices, i.e. it is proportional to $\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d$. Clearly, once we assume that $h_{abcd}\sim\mathrm{str}\,\lambda_a\lambda_b\lambda_c\lambda_d$, the abelian result (2.25) generalises to the non-abelian case. We want to go a little further and show that this is not only sufficient but actually necessary for the desired rearrangement to hold. We begin with the Fierz identity (A.6) with $\chi \to \chi^a$, $\psi \to D_\nu \chi^b$, $\lambda \to D_\mu \chi^c$ and $\varphi \to \chi^d$:

$$\begin{aligned}\overline{\chi}^a \gamma_{(\mu} D^\nu \chi^b \; D^\mu \overline{\chi}^c \gamma_{\nu)} \chi^d = & -\frac{1}{8} \overline{\chi}^a \gamma_{(\mu} \chi^d \; D^\mu \overline{\chi}^c \gamma_{\nu)} D^\nu \chi^b + \frac{1}{16} \overline{\chi}^a \gamma_\nu \chi^d D_\mu \overline{\chi}^c \gamma^\nu D^\mu \chi^b + \\ & + \frac{1}{16} \overline{\chi}^a \gamma^{\rho\sigma}{}_{(\mu} \chi^d D^\mu \overline{\chi}^c \gamma_{\nu)\rho\sigma} D^\nu \chi^b - \\ & - \frac{1}{96} \overline{\chi}^a \gamma^{\rho\sigma\lambda} \chi^d D_\mu \overline{\chi}^c \gamma_{\rho\sigma\lambda} D^\mu \chi^b - \\ & - \frac{1}{384} \overline{\chi}^a \gamma^{\rho\sigma\lambda\kappa}{}_{(\mu} \chi^d \; D^\mu \overline{\chi}^c \gamma_{\nu)\rho\sigma\lambda\kappa} D^\nu \chi^b + \\ & + \frac{1}{3840} \overline{\chi}^a \gamma^{\mu\rho\sigma\lambda\kappa} \chi^d \; D_\nu \overline{\chi}^c \gamma_{\mu\rho\sigma\lambda\kappa} D^\nu \chi^b \,. \end{aligned} \tag{2.54}$$

The first term on the r.h.s. vanishes on-shell as does the l.h.s. when μ and ν are exchanged. The other terms can be simplified using the on-shell condition and partial integration so that

$$\begin{aligned}\overline{\chi}^a \gamma_\mu D_\nu \chi^b \; D^\mu \overline{\chi}^c \gamma^\nu \chi^d \simeq & \frac{1}{2} \left(\overline{\chi}^{(a} \gamma_\rho D_\sigma \chi^{d)} - \overline{\chi}^{(a} \gamma_\sigma D_\rho \chi^{d)} \right) \overline{\chi}^c \gamma^\sigma D^\rho \chi^b - \\ & - \frac{1}{4} \overline{\chi}^{[a} \gamma_\sigma D_\rho \chi^{d]} \; \overline{\chi}^c \gamma^\sigma D^\rho \chi^b + \\ & + \frac{1}{24} \overline{\chi}^a \gamma_{\rho\sigma\lambda} D_\kappa \chi^d \overline{\chi}^c \gamma^{\rho\sigma\lambda} D^\kappa \chi^b - \frac{1}{8} \overline{\chi}^{[a} \gamma_{\rho\sigma\nu} D_\mu \chi^{d]} \overline{\chi}^c \gamma^{\rho\sigma\mu} D_\nu \chi^b - \\ & - \frac{1}{960} \overline{\chi}^{[a} \gamma_{\mu\rho\sigma\lambda\kappa} D_\nu \chi^{d]} \overline{\chi}^c \gamma^{\mu\rho\sigma\lambda\kappa} D^\nu \chi^b \,. \end{aligned} \tag{2.55}$$

While the first and second terms exhibits the desired form and the third term can be dealt with by using again the Fierz identity (A.5), the fourth term is as troublesome (if not more) as the initial $\overline{\chi}^a \gamma_\mu D_\nu \chi^b \; D^\mu \overline{\chi}^c \gamma^\nu \chi^d$ we want to get rid of. However it comes antisymmetrised in a and d, as does the fifth term, so that the symmetric part of eq. (2.55) simply reduces to, using (A.5) again,

$$\overline{\chi}^a \gamma_\mu D_\nu \chi^b \, \overline{\chi}^c \gamma^\nu D^\mu \chi^d + (a \leftrightarrow d) \simeq \frac{2}{3} \, \overline{\chi}^a \gamma_\mu D_\nu \chi^b \, \overline{\chi}^c \gamma^\mu D^\nu \chi^d + (a \leftrightarrow d) \,. \tag{2.56}$$

Thus we will get rid of the troublesome term and be able to use eq. (2.56) provided h_{abcd} is symmetric under exchange of a and d which we assume from now on. But as noted above, this implies that h_{abcd} is completely symmetric in all its indices:

$$h_{abcd} = h \,\mathrm{str}\, \lambda_a \lambda_b \lambda_c \lambda_d \,. \tag{2.57}$$

Hence, the contribution to the amplitude of the second term in $\mathcal{L}_{4\mathrm{f}}$ is

$$A_4^{4\mathrm{f}}|_{h-\mathrm{terms}} = -4i\tilde{\alpha}'^2 h \;\mathrm{str}\, \lambda_a \lambda_b \lambda_c \lambda_d \left(s \, \overline{u}_1 \gamma_\mu u_4 \overline{u}_2 \gamma^\mu u_3 - u \; \overline{u}_1 \gamma_\mu u_2 \overline{u}_3 \gamma^\mu u_4 \right) . \tag{2.58}$$

Matching the sum of both contributions (2.51) and (2.58) to the order α'^2 four fermion string amplitude (recall that $\tilde{\alpha}' = 2\pi g \alpha'$) we get the condition

$$2g_1 + g_3 + 2h = -\frac{1}{8} \,. \tag{2.59}$$

As a consistency check, we note that the *abelian* Born-Infeld action (2.21) corresponds to $g_1 = g_3 = 1/8$ and $h = -1/4$ which do satisfy this relation. With this in mind we parametrise

$$g_1 = \frac{1}{8}\left(1 + \delta g + \frac{4}{3}\delta h\right) , \qquad g_3 = \frac{1}{8}\left(1 - 2\delta g + \frac{4}{3}\delta h\right) , \qquad h = -\frac{1}{4}(1 + \delta h) . \tag{2.60}$$

Similarly to what happened for the 2 boson / 2 fermion amplitude, matching of the 4 fermion amplitude does not completely determine the U(N) tensor structure. Explicitly, we have found that the string 4 fermion amplitude is reproduced for any of the following interactions with arbitrary δg and δh:

$$\begin{aligned}\mathcal{L}_{4\mathrm{f}} = \mathrm{str}\left(\frac{\tilde{\alpha}'^2}{8}\left(1 + \delta g + \frac{4}{3}\delta h\right)\overline{\chi}\gamma^\mu D^\nu \chi \overline{\chi}\gamma_\mu D_\nu \chi - \frac{\tilde{\alpha}'^2}{4}(1 + \delta h)\overline{\chi}\gamma^\mu D^\nu \chi\overline{\chi}\gamma_\nu D_\mu \chi\right) - \\ - \frac{\tilde{\alpha}'^2}{32}\delta g \; d_{ade} d_{bce} \overline{\chi}^a \gamma^\mu D^\nu \chi^b \, \overline{\chi}^c \gamma_\mu D_\nu \chi^d .\end{aligned} \tag{2.61}$$

Note that the parameter δh does not reflect a lack of knowledge of the precise form of the action, but it only expresses the freedom to use the "Fierz" identity (2.56) to write the same term in two different ways: we may choose any δh and still have the same action. The free parameter δg corresponds to a contribution that vanishes on-shell. It is proportional to (using (A.11))

$$(f_{ace} f_{bde} + f_{abe} f_{cde})\overline{\chi}^a \gamma^\mu D^\nu \chi^b \, \overline{\chi}^c \gamma_\mu D_\nu \chi^d . \tag{2.62}$$

The first term we rewrite, using a Fierz transformation and contracting γ-matrices, in the form

$$-\frac{1}{2} f_{ace} f_{bde} \overline{\chi}^c \gamma^\mu \chi^a D^\nu \overline{\chi}^b \gamma_\mu D_\nu \chi^d . \tag{2.63}$$

In the second term we do a partial integration, obtaining (up to terms that vanish on-shell)

$$-\frac{1}{2} f_{abe} f_{cde} \overline{\chi}^a \gamma^\mu \chi^b D^\nu \overline{\chi}^c \gamma_\mu D_\nu \chi^d . \tag{2.64}$$

The two expressions are now in the same form, and can be seen to cancel after renaming the indices. Therefore also the four-fermion terms in the effective action are determined (up to contributions that vanish on-shell) by the corresponding string amplitude.

2.4 The string effective action

Finally we are in a position to collect our results and give the effective action up to and including all order α'^2 terms, bosonic, fermionic and mixed.[2] Without loss of

[2]As already emphasized, we have nothing to say about a possible term $\sim \alpha'^2 F \overline{\chi}\gamma\chi\,\overline{\chi}\gamma\chi$ which would only show up in a five-point amplitude.

generality we choose $\delta h = 0$. Then the effective action reads:

$$\begin{aligned}\mathcal{L}_{\text{string}} = \text{str}\bigg(& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\chi}\gamma^\mu D_\mu\chi + \frac{\tilde{\alpha}'^2}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{\tilde{\alpha}'^2}{32}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \\ & + i\frac{\tilde{\alpha}'^2}{4}\overline{\chi}\gamma_\mu D_\nu\chi F^{\mu\rho}F_\rho{}^\nu - i\frac{\tilde{\alpha}'^2}{8}\overline{\chi}\gamma_{\mu\nu\rho}D_\sigma\chi F^{\mu\nu}F^{\rho\sigma} + \\ & + \frac{\tilde{\alpha}'^2}{8}\overline{\chi}\gamma^\mu D^\nu\chi\,\overline{\chi}\gamma_\mu D_\nu\chi - \frac{\tilde{\alpha}'^2}{4}\overline{\chi}\gamma^\mu D^\nu\chi\,\overline{\chi}\gamma_\nu D_\mu\chi\bigg) + \mathcal{O}(\alpha'^3 g^2, \alpha'^2 g^3)\,. \end{aligned} \tag{2.65}$$

Obviously, in the abelian limit this reduces to the standard abelian Born-Infeld action (2.24) after the field redefinition (2.23). But the comparison with the Born-Infeld action as obtained by expanding the determinant goes further. Indeed, this non-abelian string effective action coincides with the result of the following manipulation: Take the abelian Born-Infeld action and expand it up to and including order α'^2. Make the field redefinition to eliminate the order α' term, and drop all "on-shell" terms $\sim \tilde{\alpha}'^2 \not{\partial}\chi$. This gives (2.24). Only then proceed to the obvious non-abelian generalisation and take a symmetrised trace. As noted above, this is not the same as taking the symmetrised trace before the field redefinition. This correct procedure might be called the modified symmetrised trace prescription. Note that it is unlikely that some sort of modified symmetrised trace prescription continues to hold at higher orders in α'.

At this point it is useful to compare with the results of [8]. There, the $d = 10$ super Yang-Mills action through order α'^2 was also determined by requiring linear supersymmetry. The claim is that the result is essentially unique. While the Lorentz structure is completely fixed there remains some small freedom in the adjoint structure, but again the only choice consistent with string theory turns out to be a symmetrised trace. If this uniqueness claim is correct, the action given in ref. [8] and our string effective action (2.65) must coincide (up to on-shell terms and total derivatives). As we will now show, this is indeed the case. When using the same normalisation as ours, the action of ref. [8] becomes

$$\begin{aligned}\mathcal{L}_{\text{susy}} = \text{str}\bigg(& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\chi}\gamma^\mu D_\mu\chi + \frac{\tilde{\alpha}'^2}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{\tilde{\alpha}'^2}{32}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \\ & + i\frac{\tilde{\alpha}'^2}{4}\overline{\chi}\gamma_\mu D_\nu\chi F^{\mu\rho}F_\rho{}^\nu - i\frac{\tilde{\alpha}'^2}{8}\overline{\chi}\gamma_{\mu\nu\rho}D_\sigma\chi F^{\mu\nu}F^{\rho\sigma} - \\ & - \frac{1}{1440}\,\tilde{\alpha}'^2\overline{\chi}\gamma^{\mu\nu\rho}\chi D_\sigma\overline{\chi}\,\gamma_{\mu\nu\rho}D^\sigma\chi - \frac{3}{80}\,\tilde{\alpha}'^2\overline{\chi}\gamma^{\mu\nu\rho}\chi\, D_\mu\overline{\chi}\gamma_\nu D_\rho\chi - \\ & - \frac{7}{480}\tilde{\alpha}'^2 g\, F^{\mu\nu}\overline{\chi}\gamma_{\mu\nu\rho}\chi\{\overline{\chi},\gamma^\rho\chi\} + \frac{1}{2880}\tilde{\alpha}'^2 g\, F^{\mu\nu}\overline{\chi}\gamma^{\rho\sigma\tau}\chi\{\overline{\chi},\gamma_{\mu\nu\rho\sigma\tau}\chi\}\bigg) + \\ & + \mathcal{O}(\tilde{\alpha}'^3)\,, \end{aligned} \tag{2.66}$$

The first two lines of eqs. (2.66) and eq. (2.65) agree. It is clear that a direct (string) calculation of the last line of eq. (2.66) which is of order $\alpha'^2 g^3$ would require the calculation of five point scattering amplitudes, so we have nothing to say about it here. What remains is the third line of (2.66) which has to be compared with the last line in (2.65). Using the following identities (cf. (2.56) and (A.5))

$$\begin{aligned}
\mathrm{str}\,\overline{\chi}\gamma^{\mu}D^{\nu}\chi\,\overline{\chi}\gamma_{\nu}D_{\mu}\chi &\simeq \frac{2}{3}\,\mathrm{str}\,\overline{\chi}\gamma^{\mu}D^{\nu}\chi\,\overline{\chi}\gamma_{\mu}D_{\nu}\chi \\
\mathrm{str}\,\overline{\chi}\gamma^{\mu\nu\rho}\chi D_{\sigma}\overline{\chi}\,\gamma_{\mu\nu\rho}D^{\sigma}\chi &= 24\;\,\mathrm{str}\,\overline{\chi}\gamma^{\mu}D^{\nu}\chi\,\overline{\chi}\gamma_{\mu}D_{\nu}\chi \\
\mathrm{str}\,\overline{\chi}\gamma^{\mu\nu\rho}\chi\,D_{\mu}\overline{\chi}\gamma_{\nu}D_{\rho}\chi &\simeq \frac{2}{3}\,\mathrm{str}\,\overline{\chi}\gamma^{\mu}D^{\nu}\chi\,\overline{\chi}\gamma_{\mu}D_{\nu}\chi\,,
\end{aligned} \tag{2.67}$$

we find that the third line of eq. (2.66) agrees with the third line of eq. (2.65), provided

$$-\frac{1}{1440}\times 24-\frac{3}{80}\times\frac{2}{3}=\frac{1}{8}-\,-\frac{1}{4}\times\frac{2}{3}\,, \tag{2.68}$$

which indeed is true.

In [8], the presence of a non-linear supersymmetry of the action (2.66) was established as well. This provided strong consistency checks on various terms although the values of the coefficients of the four-fermion terms are insensitive to this. Indeed, one easily checks that the variation of these two terms under the non-linear supersymmetry result in expressions proportional to equations of motion. Happily, as just checked, these terms are precisely equivalent to the four-fermion terms in the string effective action, which now provides an independent check.

3. Kappa-symmetry

The purpose of this section is to compare the results for the effective action obtained in section 2 with the results that follow from the requirement of κ-symmetry. In the abelian case, κ-symmetry has led, in the limit of constant F, to exact answers for D-branes in a flat [2] as well as a curved [18] background.

Another reason to reconsider the results obtained in [9], is the recent claim [8], that supersymmetric Yang-Mills theory in $d = 10$, to order α'^2, must contain a symmetric trace of the Yang-Mills generators. According to [8] any deviation from the symmetric trace must be trivial, in the sense that it can be removed by a field redefinition.

The results of [9] indicate that, as far as the terms bilinear in the fermions are concerned, a nontrivial deviation from the symmetric trace does occur. Since quartic fermions were not considered in [9], we will disregard them in this section. The results of [9] are based on the assumption that a particular non-abelian version of κ-symmetry exists. This non-abelian κ-symmetry automatically leads to linear and nonlinear supersymmetries after κ-gauge fixing. The non-abelian κ-symmetry proposal of [9] was only established at order F^2 in the variation. This implies that after κ-gauge fixing the linear supersymmetry has only been established for the F^2

terms in the action but not for the $\alpha'^2 F^4$ terms (there are no F^3 terms). The action of [9] also contains terms which are of the form $\alpha'^2\bar{\theta}\partial\theta F^2$. These terms are needed to realize the nonlinear supersymmetry at order F^2. On the other hand, the linear supersymmetry calculation of [8] was performed up to order α'^2 in the variation. This fixes the $\alpha'^2 F^4$ terms in the action which, by linear supersymmetry, are connected to the $\alpha'^2\bar{\theta}\partial\theta F^2$ terms.

The apparent contradiction between [8] and [9] is that the linear supersymmetry calculation of [8] leads to a symmetric trace prescription of the $\alpha'^2\bar{\theta}\partial\theta F^2$ terms in the effective action whereas the κ-symmetry calculation of [9] shows that these terms do not satisfy the symmetric trace prescription.

We should keep in mind that since κ-symmetry has only been established up to order F^2 terms in the variation, we have no guarantee that we can proceed to higher orders. Indeed, the results of [8] indicate that proceeding with the κ-symmetry calculation to the next order might be problematic.

Strictly speaking there are two possible situations:

1. It is possible that after redefinitions the $\alpha'^2\bar{\theta}\partial\theta F^2$ terms of [9] do become a symmetric trace, in which case the result agrees with [8].

2. If it is not a symmetric trace, under any field-redefinition, then, assuming that the conclusion of [8] is correct, κ-symmetry must fail at the next order.

We will show in the remainder of this section that the first possibility does not apply. There are no field redefinitions under which all terms in the action of [9] can be written as a symmetrised trace. We are left with the second possibility and, indeed, we will show that κ-symmetry fails at order F^3 in the variation. The consequences of this will be discussed in the next section.

3.1 kappa-invariant action

It is convenient to first reformulate the results of [9] in the form obtained after making the field redefinitions discussed in section 2. For the κ-symmetric formulation these redefinitions take the form:

$$\begin{aligned} \bar{\theta}^a &\to \bar{\theta}'^{\,a} - \frac{1}{8}\tilde{\alpha}' d^{abc}\,\bar{\theta}'^{\,b}\sigma_3\gamma\cdot F^c P_- \,, \\ A^a_\mu &\to A'^{\,a}_\mu - \frac{i}{8}\tilde{\alpha}' d^{abc}\,\bar{\theta}'^{\,b}\gamma_{11}(i\sigma_2)\gamma_\mu\theta'^{\,c}\,. \end{aligned} \tag{3.1}$$

Here we use the following notation:

$$P_\pm = \frac{1}{2}(1 \pm \gamma_{11}\sigma_1)\,. \tag{3.2}$$

These are projection operators, and satisfy

$$P_\pm P_\pm = P_\pm\,,\ \ P_- P_+ = 0\,, \qquad P_-\sigma_3 = \sigma_3 P_+\,. \tag{3.3}$$

After this redefinition the action is

$$
\begin{aligned}
\mathcal{L} = {} & i\bar{\theta}^a P_-\gamma^\mu \mathcal{D}_\mu \theta^a - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \\
& + \frac{i}{8}\tilde{\alpha}'^2 d^{ae(c} d^{d)be} \bar{\theta}^a P_- \gamma_\mu \mathcal{D}_\nu \theta^b T^{\mu\nu\, cd} - \\
& - \frac{i}{16}\tilde{\alpha}'^2 d^{ae[c} d^{d]be} \bar{\theta}^a P_- \gamma_{\mu\nu\rho} \{ \mathcal{D}^\rho \theta^b F^{\mu\sigma\, c} F_\sigma{}^{\nu\, d} - \mathcal{D}_\sigma \theta^b F^{\mu\nu\, c} F^{\rho\sigma\, d} \} + \\
& + \frac{i}{64}\tilde{\alpha}'^2 d^{ace} d^{bde} \bar{\theta}^a P_- \gamma_{\mu\nu\rho\sigma\tau} \mathcal{D}_\tau \theta^b F^{\mu\nu\, c} F^{\rho\sigma\, d} \,,
\end{aligned}
\tag{3.4}
$$

where $T^{\mu\nu\, cd}$ is the nonabelian generalization of the energy-momentum tensor:

$$
T^{\mu\nu\, cd} = F^{\mu\rho\,(c} F_\rho{}^{d)\nu} + \frac{1}{4}\eta^{\mu\nu} F^c_{\rho\sigma} F^{\rho\sigma\, d} \,. \tag{3.5}
$$

It is invariant under the following κ-symmetry transformations:

$$
\delta\bar{\theta}^a = \bar{\eta}^a - \bar{\epsilon}^a + \frac{1}{8}\tilde{\alpha}' d^{abc}\, (\bar{\eta}^b - \bar{\epsilon}^b) \sigma_3 \gamma \cdot F^c P_- \,, \tag{3.6}
$$

$$
\begin{aligned}
\delta A^a_\mu = {} & \frac{i}{2}\tilde{\alpha}' d^{abc}\, \bar{\eta}^b P_- \sigma_3 \gamma_\mu \theta^c + \frac{i}{2}\tilde{\alpha}' d^{abc}\, \bar{\epsilon}^b P_+ \sigma_3 \gamma_\mu \theta^c + \\
& + \frac{i}{8}\tilde{\alpha}'^2 d^{ae(c} d^{d)be} \bar{\epsilon}^b P_- \gamma_\rho \theta^c F^\rho{}_\mu{}^d + \\
& + \frac{i}{16}\tilde{\alpha}'^2 d^{ae[c} d^{d]be} \bar{\epsilon}^b P_- \gamma_{\mu\nu\rho} \theta^c F^{\nu\rho\, d} + \\
& + \frac{i}{32}\tilde{\alpha}'^2 (3 d^{ace} d^{bde} + d^{abe} d^{cde})\, \bar{\eta}^b (P_+ + P_-) \gamma_k \theta^c F^k{}_i{}^d + \\
& + \frac{i}{16}\tilde{\alpha}'^2 d^{ae[c} d^{d]be} \bar{\eta}^b (P_+ - P_-) \gamma_\rho \theta^c F^\rho{}_\mu{}^d + \\
& + \frac{i}{16}\tilde{\alpha}'^2 d^{ace} d^{bde} \bar{\eta}^b P_+ \gamma_{\mu\nu\rho} \theta^c F^{\nu\rho\, d} + \\
& + \frac{i}{16}\tilde{\alpha}'^2 d^{ae[c} d^{d]be} \bar{\eta}^b P_- \gamma_{\mu\nu\rho} \theta^c F^{\nu\rho\, d} \,,
\end{aligned}
\tag{3.7}
$$

where the parameter η^a is of the form

$$
\bar{\eta}^a = \bar{\kappa}^b (\delta^{ab} + \Gamma^{ab}) \,. \tag{3.8}
$$

The matrix Γ must square to one, and can be reconstructed in the present basis from the results given in [9]. The parameter ϵ^a is constant, and must satisfy $f^{abc}\epsilon^c = 0$.

3.2 Gauge fixing and supersymmetry

Gauge-fixing follows the same lines as discussed in [9]. The κ-symmetry is gauge-fixed by setting $\theta_2^A = 0$, and the remaining symmetries are linear and nonlinear supersymmetry. We will present only the results. After gauge-fixing the action reads:

$$
\begin{aligned}
\mathcal{L} = {} & \frac{i}{2}\bar{\chi}^a\gamma^\mu\mathcal{D}_\mu\chi^a - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \\
& + \frac{i}{16}\tilde{\alpha}'^2 d^{ae(c}d^{d)be}\bar{\chi}^a\gamma_\mu\mathcal{D}_\nu\chi^b T^{\mu\nu\,cd} - \\
& - \frac{i}{32}\tilde{\alpha}'^2 d^{ae[c}d^{d]be}\bar{\chi}^a\gamma_{\mu\nu\rho}\{\mathcal{D}^\rho\chi^b F^{\mu\sigma\,c}F_\sigma{}^{\nu\,d} - \mathcal{D}_\sigma\chi^b F^{\mu\nu\,c}F^{\rho\sigma\,d}\} + \\
& + \frac{i}{128}\tilde{\alpha}'^2 d^{ace}d^{bde}\bar{\chi}^a\gamma_{\mu\nu\rho\sigma\tau}\mathcal{D}_\tau\chi^b F^{\mu\nu\,c}F^{\rho\sigma\,d}\,.
\end{aligned}
\tag{3.9}
$$

The transformation rules under supersymmetry simplify because of the condition $f^{abc}\epsilon^c = 0$. This means we can choose a basis in the U(N) Lie-algebra such that only one ϵ, corresponding to the U(1) direction, remains. Setting $a = 0$ for the U(1) direction, we then use $d^{ab0} = \delta^{ab}$ (up to a constant, which we absorb into the normalisation of ϵ). The transformation rules then take on the following form:

$$
\delta\bar{\chi}^a = -(\bar{\epsilon}_1 + \bar{\epsilon}_2)\delta^{a0} - \frac{1}{8}\tilde{\alpha}'(\bar{\epsilon}_1 - \bar{\epsilon}_2)\gamma\cdot F^a\,, \tag{3.10}
$$

$$
\delta A^a_\mu = +\frac{i}{4}\tilde{\alpha}'(\bar{\epsilon}_1 - \bar{\epsilon}_2)\gamma_\mu\chi^a + \frac{i}{8}\tilde{\alpha}'^2 d^{acd}(\bar{\epsilon}_1 + \bar{\epsilon}_2)\gamma_\rho\chi^c F^\rho{}_\mu{}^d\,. \tag{3.11}
$$

The algebra of the linear supersymmetry is as usual. The nonlinear supersymmetry gives a covariant translation on A (the same as for linear supersymmetry). To see this on χ would require the presence of higher-order fermions in the transformation rule of χ, but these have not been determined. The algebra of linear with nonlinear supersymmetry gives a constant shift on the U(1) vector. On χ this commutator also requires higher-order fermion contributions.

The action (3.9) can be simplified somewhat by redefining χ^a with F^2-dependent terms. This gives:

$$
\begin{aligned}
\mathcal{L} = {} & \frac{i}{2}\bar{\chi}^a\gamma^\mu\mathcal{D}_\mu\chi^a - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \frac{i}{16}\tilde{\alpha}'^2 d^{ace}d^{bde}\bar{\chi}^a\gamma_\mu\mathcal{D}_\nu\chi^b F^{\mu\rho\,c}F_\rho{}^{\nu\,d} - \\
& - \frac{i}{32}\tilde{\alpha}'^2 d^{ade}d^{bce}\bar{\chi}^a\gamma_{\mu\nu\rho}\mathcal{D}_\sigma\chi^b F^{\mu\nu\,c}F^{\rho\sigma\,d}\,.
\end{aligned}
\tag{3.12}
$$

In this form the result can be most easily compared with the results of section 2. Note that (3.12) is not a symmetric trace and therefore it differs from the action (2.65) we found in section 2. This is an aspect of the κ-symmetric formulation which now is seen to be independent of the redefinition we performed. Of course, in the abelian limit ($d^{abc} \to 2$), the action (3.12) should coincide with the action (2.65) of section 2, as it does.

At order α'^2 we can only check the nonlinear supersymmetry, and it is indeed valid. For the linear supersymmetry at order α'^2 we need also the F^4 term. In fact, long ago, in [7], it was shown that the following action (ignoring quartic fermions),

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which has a symmetric trace, is invariant under linear supersymmetry:

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{12}(d^{abe}d^{cde} + d^{ade}d^{bce} + d^{ace}d^{bde})\tilde{\alpha}'^2 \times \\
&\times \Bigg(\frac{1}{8} F^a_{\mu\nu} F^{\nu\rho\, b} F^c_{\rho\sigma} F^{\sigma\mu\, d} - \frac{1}{32} F^a_{\mu\nu} F^{\mu\nu\, b} F^c_{\rho\sigma} F^{\rho\sigma\, d} + \\
&\quad + \frac{i}{4}\bar{\chi}^a \gamma_\mu \mathcal{D}_\nu \chi^b T^{\mu\nu\, cd} - \frac{i}{8}\bar{\chi}^a \gamma_{\mu\nu\rho}\{\mathcal{D}^\rho \chi^b F^{\mu\sigma\, c} F_\sigma{}^{\nu\, d} - \mathcal{D}_\sigma \chi^b F^{\mu\nu\, c} F^{\rho\sigma\, d}\} + \\
&\quad + \frac{i}{32}\bar{\chi}^a \gamma_{\mu\nu\rho\sigma\tau} \mathcal{D}_\tau \chi^b F^{\mu\nu\, c} F^{\rho\sigma\, d} \Bigg) .
\end{aligned}
\tag{3.13}
$$

Performing an analogous redefinition of χ^a with F^2-dependent terms, as above, this can be rewritten as

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{12}(d^{abe}d^{cde} + d^{ade}d^{bce} + d^{ace}d^{bde})\tilde{\alpha}'^2 \times \\
&\times \Bigg(\frac{1}{8} F^a_{\mu\nu} F^{\nu\rho\, b} F^c_{\rho\sigma} F^{\sigma\mu\, d} - \frac{1}{32} F^a_{\mu\nu} F^{\mu\nu\, b} F^c_{\rho\sigma} F^{\rho\sigma\, d} + \\
&\quad + \frac{i}{4}\bar{\chi}^a \gamma_\mu \mathcal{D}_\nu \chi^b T^{\mu\nu\, cd} - \frac{i}{8}\bar{\chi}^a \gamma_{\mu\nu\rho} \mathcal{D}_\sigma \chi^b F^{\mu\nu\, c} F^{\rho\sigma\, d} \Bigg) .
\end{aligned}
\tag{3.14}
$$

We now want to check that our action (3.12) can be made invariant under linear supersymmetry after adding the well-known STr F^4 terms predicted by string theory. The simplest way to check this is to add to (3.12) a symmetric trace F^4 term, with the correct normalization, and then to subtract the result from (3.14). This difference, $\mathcal{L}_{\text{rest}}$, should then also be supersymmetric. This can only happen if the variation of this difference can be cancelled by new order α'^2-variations of the fields χ and A. This requires that all terms in the variation can be rewritten in terms of the (lowest order) equations of motion of these fields. For this analysis it is of course crucial that there are no other parts of the action which could interfere with this calculation, such as higher-derivative terms. We have verified that to this order higher-derivative contributions can always be reexpressed in terms of lowest-order equations of motions, and can be eliminated by field redefinitions.

The variation of $\mathcal{L}_{\text{rest}}$ under linear supersymmetry, in which case $\epsilon \equiv \epsilon_1 - \epsilon_2$, is

$$
\begin{aligned}
\delta\mathcal{L}_{\text{rest}} = \tilde{\alpha}'^3 \Bigg(&-\frac{i}{16}\bar{\epsilon}\gamma_{\rho\sigma}\gamma_\mu \mathcal{D}_\nu \chi^a F^{\rho\sigma\, b} F^\mu{}_\tau{}^c F^{\tau\nu\, d} \, (P-Q)^{abcd} + \\
&+ \frac{i}{32}\bar{\epsilon}\gamma_{\rho\sigma}\gamma_{\mu\nu\tau} \mathcal{D}_\lambda \chi^a F^{\rho\sigma\, b} F^{\mu\nu\, c} F^{\tau\lambda\, d} \, (P+Q)^{abcd} \Bigg) .
\end{aligned}
\tag{3.15}
$$

where the tensors P and Q have been defined in the appendix. To analyze this variation, it is convenient to multiply all γ-matrices together in terms of a$\gamma^{(5)}$, a $\gamma^{(3)}$, and a $\gamma^{(1)}$. Using the symmetry properties of P and Q it is not very complicated to show that the $\gamma^{(5)}$ contribution can be written in terms of equations of motion.

However, this analysis fails at the level of the $\gamma^{(3)}$ terms. We found that for certain dimensions lower than ten (in particular $d = 3$) the $\gamma^{(3)}$-terms can also be rewritten in terms of equations of motion, but in the general case, and in particular in $d = 10$, this does not work. For $d = 3$ the $\gamma^{(1)}$-terms still give problems, which can however be resolved by adding F^4-terms which are not a symmetric trace. For $d = 10$ we conclude that κ-symmetry fails at this order.

4. Conclusions

In this paper we have determined the string effective action from the four-point string scattering amplitudes, including all fermionic terms through order $\alpha'^2 g^2$. We have also refined the determination of the κ-symmetric action of [9] by proceeding in a way which yields no order α' term from the beginning, so this corresponds to the situation after the field redefinition. The two results do not coincide. While κ-symmetry might be desirable, it is not a sacred principle. On the other hand, the effective action (2.65) we obtained by matching string amplitudes really is the true string effective action. As repeatedly mentioned, its order α'^2 terms are only determined up to on-shell terms, but this is precisely the freedom we have to perform further field redefinitions of order α'^2. In ref. [8] a super Yang-Mills action through order α'^2 including all fermionic terms was also determined recently by requiring linear supersymmetry. The claim of [8] is that the result is essentially unique and we have shown that it coincides with the string effective action we have determined. For completenes we give here the result where we have rewritten the quartic fermions as a single term, using the identities (2.67):

$$\begin{aligned}\mathcal{L}_{\text{string}} = \operatorname{str}\Big(& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\chi}\gamma^\mu D_\mu\chi + \frac{\tilde{\alpha}'^2}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{\tilde{\alpha}'^2}{32}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \\ & + i\frac{\tilde{\alpha}'^2}{4}\overline{\chi}\gamma_\mu D_\nu\chi F^{\mu\rho}F_\rho{}^\nu - i\frac{\tilde{\alpha}'^2}{8}\overline{\chi}\gamma_{\mu\nu\rho}D_\sigma\chi F^{\mu\nu}F^{\rho\sigma} - \\ & - \frac{\tilde{\alpha}'^2}{24}\overline{\chi}\gamma^\mu D^\nu\chi\,\overline{\chi}\gamma_\mu D_\nu\chi\Big) + \mathcal{O}(\alpha'^3 g^2, \alpha'^2 g^3)\,. \end{aligned} \tag{4.1}$$

We have omitted the $F\chi^4$ term since it is $\mathcal{O}(\alpha'^2 g^3)$ and would only show up in the calculation of the five-point amplitude.

We can safely conclude that the symmetrised trace prescription for the non-abelian Born-Infeld action holds through order α'^2, including *all* fermionic and derivative terms. As we pointed out in section 2.4, one should be careful with the field redefinitions. The redefinitions should be done before implementing the symmetrised trace! We also stress that the present conclusion does not imply that the symmetrised trace prescription will continue to hold at higher orders. In fact a closer investigation of the α'-expansion of the string scattering amplitudes [20] indicates that the symmetrised trace prescription will fail beyond order α'^2.

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Finally, we found that κ-symmetry cannot be extended to the order F^3 in the variation. On the other hand, the fact that the effective action through order α'^2 shows both a linear and a non-linear supersymmetry is indicative for the existence of an underlying κ-invariant formulation. The work of [9] was based on a non-abelian κ-symmetry, under which all fermions transform, such that the κ-parameter is also in the adjoint representation of the Yang-Mills group. It may be that this approach has been too ambitious, and that only a single κ-symmetry can be realised. It is also conceivable that the approach of [9] was not ambitious enough and, maybe, besides nonabelian κ-transformations, it is also required to introduce some kind of non-abelian diffeomorphisms on the worldvolume. Clearly more thought is required before κ-symmetry, in this context, is finally put to rest.

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A. Conventions and useful identities

In this appendix we gather some conventions and identities we use.

$$\textit{Kinematics:}\quad s=(k_1+k_2)^2\,,\qquad t=(k_1+k_3)^2\,,\qquad u=(k_1+k_4)^2 \tag{A.1}$$

with all momenta incoming and signature $(+,-,\ldots,-)$. Since all our states are massless we have $s+t+u=0$.

Spinors: The Clifford algebra is $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$, i.e. $(\gamma^0)^2=+1$. Antisymmetric products of γ-matrices are defined with weight 1: $\gamma_{\mu\nu}=\frac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$ etc. Often used identities are

$$\begin{aligned}
\gamma_{\mu\nu\rho} &= \gamma_\mu\gamma_\nu\gamma_\rho-\gamma_\mu\eta_{\nu\rho}+\gamma_\nu\eta_{\mu\rho}-\gamma_\rho\eta_{\mu\nu}\\
\gamma_\mu\gamma_{\nu\rho} &= \gamma_{\mu\nu\rho}+\gamma_\rho\eta_{\mu\nu}-\gamma_\nu\eta_{\mu\rho}\\
\gamma_{\nu\rho}\gamma_\mu &= \gamma_{\nu\rho\mu}+\gamma_\nu\eta_{\mu\rho}-\gamma_\rho\eta_{\mu\nu}\,.
\end{aligned} \tag{A.2}$$

The ten-dimensional spinors are 16-component Majorana-Weyl spinors and satisfy various identities. In particular, due to the Weyl property $\overline{\chi}_1\gamma_{\mu_1\ldots\mu_p}\chi_2=0$ for all even p, and the expressions with $p>5$ are related to those with $10-p<5$. Due to the Majorana property anticommuting spinor fields satisfy

$$\begin{aligned}
\overline{\chi}_1\chi_2=\overline{\chi}_2\chi_1\,,\qquad \overline{\chi}_1\gamma_\mu\chi_2=-\overline{\chi}_2\gamma_\mu\chi_1\,,\\
\overline{\chi}_1\gamma_{\mu_1\ldots\mu_p}\chi_2=(-)^p\overline{\chi}_2\gamma_{\mu_p\ldots\mu_1}\chi_1=(-)^{p(p+1)/2}\overline{\chi}_2\gamma_{\mu_1\ldots\mu_p}\chi_1\,.
\end{aligned} \tag{A.3}$$

Note that when the anticommuting spinor fields are replaced by commuting spinor wave-functions we have the analogous identities but with an extra minus sign.

There are also various Fierz identities which can be derived from the following basic identity [19] valid for ten-dimensional Majorana-Weyl spinors (a Weyl projector is implicitly assumed to multiply the r.h.s.)

$$\psi\overline{\lambda} = -\frac{1}{16}\gamma^{\mu}(\overline{\lambda}\gamma_{\mu}\psi) + \frac{1}{96}\gamma^{\mu\nu\rho}(\overline{\lambda}\gamma_{\mu\nu\rho}\psi) - \frac{1}{3840}\gamma^{\mu\nu\rho\sigma\kappa}(\overline{\lambda}\gamma_{\mu\nu\rho\sigma\kappa}\psi) \tag{A.4}$$

from which follows

$$\overline{\chi}\gamma^{\mu}\psi\ \overline{\lambda}\gamma_{\mu}\varphi = \frac{1}{2}\overline{\chi}\gamma^{\mu}\varphi\ \overline{\lambda}\gamma_{\mu}\psi - \frac{1}{24}\overline{\chi}\gamma^{\mu\nu\rho}\varphi\ \overline{\lambda}\gamma_{\mu\nu\rho}\psi \tag{A.5}$$

as well as

$$\begin{aligned}\overline{\chi}\gamma^{(\mu}\psi\ \overline{\lambda}\gamma^{\nu)}\varphi = &-\frac{1}{8}\overline{\chi}\gamma^{(\mu}\varphi\ \overline{\lambda}\gamma^{\nu)}\psi + \frac{1}{16}\overline{\chi}\gamma^{\rho\sigma(\mu}\varphi\ \overline{\lambda}\gamma^{\nu)}{}_{\rho\sigma}\psi - \frac{1}{384}\overline{\chi}\gamma^{\rho\sigma\lambda\kappa(\mu}\varphi\ \overline{\lambda}\gamma^{\nu)}{}_{\rho\sigma\lambda\kappa}\psi + \\ &+\eta^{\mu\nu}\left[\frac{1}{16}\overline{\chi}\gamma^{\rho}\varphi\overline{\lambda}\gamma_{\rho}\psi - \frac{1}{96}\overline{\chi}\gamma^{\rho\sigma\lambda}\varphi\overline{\lambda}\gamma_{\rho\sigma\lambda}\psi + \frac{1}{3840}\overline{\chi}\gamma^{\rho\sigma\lambda\kappa\tau}\varphi\overline{\lambda}\gamma_{\rho\sigma\lambda\kappa\tau}\psi\right],\end{aligned} \tag{A.6}$$

where $(\mu\nu)$ indicates symmetrisation in μ and ν.

Gauge group, d_{abc} and f_{abc} tensors: We denote by λ_a the hermitean generators of the fundamental representation of $\mathrm{U}(N)$. The various normalisations are fixed by

$$[\lambda_a, \lambda_b] = i f_{abc}\lambda_c\,, \qquad \{\lambda_a, \lambda_b\} = d_{abc}\lambda_c\,, \qquad \mathrm{tr}\,\lambda_a\lambda_b = \delta_{ab} \tag{A.7}$$

with real structure constants f_{abc} and real d_{abc}. These definitions imply

$$\mathrm{tr}[\lambda_a, \lambda_b]\lambda_c = i f_{abc}\,, \qquad \mathrm{tr}\{\lambda_a, \lambda_b\}\lambda_c = d_{abc}\,. \tag{A.8}$$

The generators of the adjoint representation are $(T_a^{\mathrm{adj}})_{bc} = -i f_{abc}$, which is the only representation of interest to us. The covariant derivative then is

$$(D_\mu^{\mathrm{adj}})_{ac} = \delta_{ac}\partial_\mu - igA_\mu^b(T_b^{\mathrm{adj}})_{ac} = \delta_{ac}\partial_\mu + g f_{abc}A_\mu^b\,. \tag{A.9}$$

The field strength then is given by $[D_\mu, D_\nu]_{ac} = g f_{abc}F^b_{\mu\nu}$ i.e.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu\,. \tag{A.10}$$

Possible 4-index tensors on the gauge group that could arise from a single trace are of the form $d_{abe}d_{cde}$, $f_{abe}f_{cde}$ or $d_{abe}f_{cde}$. There are 12 such possible tensors, but they are related by various Jacobi identities:

$$\begin{aligned} f_{abe}f_{cde} &= d_{ace}d_{bde} - d_{ade}d_{bce} \\ d_{abe}f_{cde} + d_{bce}f_{ade} + d_{cae}f_{bde} &= 0\,. \end{aligned} \tag{A.11}$$

The first type of identities allows to express all ff tensors as dd tensors, and the second type of identities allows to express 3 among the 6 df tensors in terms of the 3 others. We may choose $\beta_1 = d_{abe}f_{cde}$, $\beta_2 = d_{cde}f_{abe}$ and $\beta_3 = d_{ade}f_{bce} - d_{bde}f_{ace}$ as independent, and use them to express the three other $\beta_4 = d_{ace}f_{bde}$, $\beta_5 = d_{bce}f_{ade}$ and $\beta_6 = d_{ade}f_{bce} + d_{bde}f_{ace}$:

$$\beta_4 = -(\beta_1 + \beta_3)/2 + \beta_2 \,, \qquad \beta_5 = -(\beta_1 - \beta_3)/2 - \beta_2 \,, \qquad \beta_6 = \beta_1 \,. \tag{A.12}$$

Then, if we expand a general tensor as

$$\begin{aligned} X_{abcd} = {} & x_1 d_{abe}d_{cde} + x_2 d_{ace}d_{bde} + x_3 d_{ade}d_{bce} + x_4 d_{abe}f_{cde} + \\ & + x_5 d_{cde}f_{abe} + x_6(d_{ade}f_{bce} - d_{bde}f_{ace}) \,, \end{aligned} \tag{A.13}$$

knowing only $X_{(ab)cd}$ will leave $x_2 - x_3$, x_5 and x_6 undetermined, while knowing $X_{abcd} + X_{badc}$ will leave x_4 and x_5 undetermined. Finally we note that

$$\mathrm{str}\, \lambda_a\lambda_b\lambda_c\lambda_d = \frac{1}{12}\left(d_{abe}d_{cde} + d_{ace}d_{bde} + d_{ade}d_{bce}\right) \,. \tag{A.14}$$

In the text we have introduced the tensors P and Q:

$$\begin{aligned} P^{abcd} &\equiv \frac{1}{24}(d^{ace}d^{bde} + d^{ade}d^{bce} - 2d^{abe}d^{cde}) = -\frac{1}{24}(f^{ace}f^{bde} + f^{ade}f^{bce}) \\ Q^{abcd} &\equiv \frac{1}{8}(d^{ace}d^{bde} - d^{ade}d^{bce}) = \frac{1}{8}f^{abe}f^{cde} \,. \end{aligned} \tag{A.15}$$

The combinations $P \pm Q$ are:

$$\begin{aligned} (P+Q)^{abcd} &= \frac{1}{12}(f^{abe}f^{cde} + f^{ade}f^{cbe}) \\ (P-Q)^{abcd} &= -\frac{1}{12}(f^{abe}f^{cde} + f^{ace}f^{bde}) \,. \end{aligned} \tag{A.16}$$

Note that $P+Q$ is symmetric in bd and ac, $P-Q$ is symmetric in bc, ad.

Feynman rules: From $\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\chi}\gamma^\mu D_\mu\chi\right)$ we read the following Feynman rules for tree amplitudes (no ghosts): the fermion propagator is $+i\delta_{ab}/\not{k}$, the gluon propagator $-i\delta_{ab}\eta_{\mu\nu}/k^2$ (any gauge dependent additional terms $\sim k_\mu$ or $\sim k_\nu$ drop out in all our amplitudes). All vertices are obtained from the relevant interaction terms with the rule $\partial_\mu \to -ik_\mu$ where the momentum k is going into the vertex.

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